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An ANOVA-type test for multiple change points

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Abstract We consider the problem of testing the null hypothesis of no change against the alternative of multiple change points in a series of independent observations. We propose an ANOVA-type test statistic and obtain its asymptotic null distribution. We also give approximations of its limiting critical values. We report the results of Monte Carlo studies conducted to compare the power of the proposed test against a number of its competitors. As illustrations we analyzed three real data sets.

Keywords Brownian bridge · Limit theorems · Monte Carlo simulations

1 Introduction

Change-point analysis has received considerable attention in the past three decades. Statistical inference for change-point analysis involves likelihood ratio, least squares, nonparametric, sequential and Bayesian methods. Change point models are of increasing use in various fields such as Climatology, Economics, Finance, Marketing, Medicine, Psychology and Quality Control. Several examples can be found in Braun et al. (2000), Andreou and Ghysels (2006) and Villarini et al. (2011).

Let $X_1, X_2, \ldots, X_n$ be independent random variables with distribution functions $F_i(\cdot) = F(\cdot - \mu_i),\ i = 1, 2, \cdots, n$, respectively, where $F(\cdot)$ is unknown. We will assume throughout this paper that $F(\cdot)$ is continuous and has a finite variance. We consider here the problem of testing the null hypothesis of no change

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$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_n \tag{1}$$

against the multiple k-change points alternative

$$\begin{aligned} H_1 \ &: \ \exists 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 1 \quad \text{such that} \\ \mu_1 &= \cdots = \mu_{[n\lambda_1]} \neq \mu_{[n\lambda_1]+1} = \cdots = \mu_{[n\lambda_2]} \\ &\neq \cdots \neq \mu_{[n\lambda_k]+1} = \cdots = \mu_n, \end{aligned} \tag{2}$$

where $[y]$ is the integer part of y.

For testing H_0 of (1) against H_1 of (2), Lombard (1987) and Aly and BuHamra (1996) proposed and studied rank tests and Aly and Bouzar (1993) proposed and studied likelihood ratio tests. Aly et al. (2003) considered the problem of testing H_0 against the ordered multiple change points alternative which corresponds to (2) when all the $\neq$ signs are replaced by $\leq$.

For additional results and references on change point analysis and its applications we refer to Zacks (1983), Bhattacharyya (1984), Csörgő and Horváth (1988a),Csörgő and Horváth (1988b), Sen (1988), Lombard (1989), Hušková and Sen (1989), Chen and Gupta (1997), Csörgő and Horváth (1997), Chib (1998), Orasch (1999), Chen and Gupta (2000), Chong (2001), Gooijer (2005), Menne and Williams (2005), Son and Kim (2005), Lavielle and Teyssière (2006), Kim (2010), Ciuperca (2011) and Döring (2011).

The rest of this paper is organized as follows. In Sect. 2, we present the proposed test and obtain its limiting distribution. In Sect. 3, we present approximations of the limiting critical values of the proposed test. Four competing tests are presented in Sect. 4. In Sect. 5, we report the results of Monte Carlo simulations to (a) simulate the limiting critical values of the proposed test and (b) compare the power of the proposed test against a number of its competitors. As illustrations we analyzed three real data sets in Sect. 6. Finally, some concluding remarks are presented in Sect. 7.

2 The proposed test

Let $\underline{s} = (0 < s_1 < \cdots < s_k < 1)$ be such that $[ns_i] \geq [ns_{i-1}]+2, \ i = 1, 2, \ldots, k+1$ with $s_0 = 0$ and $s_{k+1} = 1$. Define $d_{i,n} = [ns_i] - [ns_{i-1}], \ i = 1, 2, \cdots, k+1$. Note that $d_{i,n}$ depends on s_i and s_{i-1}, but for notation simplicity, we do not specify it in the notation of $d_{i,n}$. Let $S_0 = 0$, $S_r = \sum_{j=1}^{r} X_j, \ r = 1, 2, \ldots, n$ and $\overline{X} = \frac{1}{n} S_n$. For $i = 1, \ldots, k+1$, the mean of $X_{[ns_{i-1}]+1}, \ldots, X_{[ns_i]}$ is

$$\overline{X}_i = \frac{S_{[ns_i]} - S_{[ns_{i-1}]}}{d_{i,n}}.$$

We propose the one-way ANOVA-type test statistic

$$T_n(k) := \int \cdots \int_{\underline{s}} V_n(\underline{s}) d\underline{s}, \tag{3}$$

where

$$V_n(\underline{s}) = \delta^{-1} n^{-(k+1)} \left(\prod_{i=1}^{k+1} d_{i,n} \right) SSTr(\underline{s}), \tag{4}$$

$$\delta = Var(X_1) \tag{5}$$

and

$$SSTr(\underline{s}) = \sum_{i=1}^{k+1} d_{i,n} \left(\overline{X}_i - \overline{X} \right)^2 .$$

Theorem 2.1 *Assume that $X_1, X_2, \ldots, X_n$ are iidrv with a common continuous distribution function $F(\cdot - \mu)$ with finite variance. Then, as $n \longrightarrow \infty$,*

$$T_n(k) \xrightarrow{D} \xi_k = \frac{1}{(2k-1)!} \int_0^1 B^2(t)dt - \int_0^1 \int_0^s Q_k(t,s) B(s) B(t) dt ds, \tag{6}$$

where $\xrightarrow{D}$ means convergence in distribution, $B(\cdot)$ is a Brownian bridge and

$$Q_k(t,s) = \sum_{j=1}^{k-1} \frac{2t^{2j-1}(1-s)^{2k-2j-1}}{(2j-1)!(2k-2j-1)!} \quad \text{for } t < s.$$

Proof Let $Y_r = S_r - \frac{r}{n} S_n,\ r = 1, \ldots, n-1$. It can be shown that

$$\begin{aligned} \left\{ \prod_{i=1}^{k+1} d_{i,n} \right\} SSTr(\underline{s}) = & \sum_{i=1}^{k} \left\{ ([ns_{i+1}] - [ns_{i-1}]) \prod_{j=1, j \neq i, i+1}^{k+1} d_{j,n} \right\} Y^2_{[ns_i]} \\ & - 2 \sum_{i=1}^{k-1} \left\{ \prod_{j=1, j \neq i+1}^{k+1} d_{j,n} \right\} Y_{[ns_i]} Y_{[ns_{i+1}]}. \end{aligned} \tag{7}$$

Based on (7) and by Theorem A.1.1 of Csörgő and Horváth (1997) it can be proved that under H_0

$$\begin{aligned} V_n(\underline{s}) \xrightarrow{D} V(\underline{s}) = & \sum_{i=1}^{k} \left\{ (s_{i+1} - s_{i-1}) \prod_{j=1, j \neq i, i+1}^{k+1} \left(s_j - s_{j-1} \right) \right\} B^2(s_i) \\ & - 2 \sum_{i=1}^{k-1} \left\{ \prod_{j=1, j \neq i+1}^{k+1} \left(s_j - s_{j-1} \right) \right\} B(s_i) B(s_{i+1}) \end{aligned} \tag{8}$$

and

$$T_n(k) \xrightarrow{D} \int \cdots_{\underline{s}} \int V(\underline{s}) d\underline{s} = \xi_k. \tag{9}$$

By (8), (9) and routine but tedious computations we obtain (6). □

Often, the number of change points k is unknown. To test the null hypothesis of no change against the alternative of an unknown number of change points given some upper bound k^* on k, we may use the statistic

$$T_{n,k^*} = \max_{1 \le k \le k^*} T_n(k).$$

The limiting distribution of this test will be a subject of future work.

Next we consider the asymptotic distribution of $T_n(k)$ under the alternative hypothesis.

Theorem 2.2 *Assume that H_1 of (2) holds true. Then, as $n \longrightarrow \infty$,*

$$T_n(k) \xrightarrow{a.s.} \infty.$$

Proof Let $\lambda_0 = 0, \lambda_{k+1} = 1$ and $\underline{\lambda} = (0 < \lambda_1 < \cdots < \lambda_k < 1)$ and assume that the change points occur at $[n\lambda_i]$, $i = 1, 2, \ldots, k$. Define $\tau_i = E\left(X_{[n\lambda_i]}\right)$, $i = 1, 2, \ldots, k+1$ and $\tau = \sum_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) \tau_i$. Note that

$$SSTr(\underline{\lambda}) = \sum_{i=1}^{k+1} \left([n\lambda_i] - [n\lambda_{i-1}]\right) \left(\frac{S_{[n\lambda_i]} - S_{[n\lambda_{i-1}]}}{[n\lambda_i] - [n\lambda_{i-1}]} - \overline{X} \right)^2$$

and

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{k+1} \left([n\lambda_i] - [n\lambda_{i-1}]\right) \left(\frac{S_{[n\lambda_i]} - S_{[n\lambda_{i-1}]}}{[n\lambda_i] - [n\lambda_{i-1}]} \right).$$

By the SLLN we can show that , as $n \longrightarrow \infty$,

$$\frac{S_{[n\lambda_i]} - S_{[n\lambda_{i-1}]}}{[n\lambda_i] - [n\lambda_{i-1}]} \overset{a.s.}{=} \tau_i + o(1),$$

$$\overline{X} \overset{a.s.}{=} \tau + o(1)$$

and

$$\frac{SSTr(\underline{\lambda})}{n} \overset{a.s.}{=} \sum_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) (\tau_i - \tau)^2 + o(1).$$

Hence,

$$\frac{V_n(\underline{\lambda})}{n} \overset{a.s.}{=} \gamma + o(1),$$

where

$$\gamma = \frac{1}{\delta} \prod_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) \sum_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) (\tau_i - \tau)^2 .$$

Note that $\gamma > 0$ under H_1 of (2) and $\gamma = 0$ under H_0 of (1). Consequently, we can argue that under H_1 of (2)

$$\frac{T_n(k)}{n} \xrightarrow{a.s.} \gamma^* > 0.$$

Hence, under H_1,

$$T_n(k) \xrightarrow{a.s.} \infty.$$

□

3 The critical values of ξ_k

Let ξ_k be as defined in (6). Consider first the case of $k = 2$.

Lemma 3.1 *Let $Z_1, Z_2, \ldots$ be iid $N(0, 1)$. Then,*

$$\xi_2 \overset{D}{=} \sum_{j=1}^{\infty} \left\{ \frac{1}{6 j^2 \pi^2} - \frac{1}{j^4 \pi^4} \right\} Z_j^2. \tag{10}$$

Proof By (6),

$$\xi_2 = \frac{1}{6} \int_0^1 B^2(t) dt - \int_0^1 \int_0^1 \{\min(t, s) - ts\} B(s) B(t) dt ds. \tag{11}$$

Following Shorack and Wellner (1986) and the proof of (2.3) of Lombard (1987), the relation (10) is obtained from (11) by the substitutions

$$B(u) = \sqrt{2} \sum_{j=1}^{\infty} \frac{Z_j}{j\pi} \sin(j\pi u), \quad 0 \le u \le 1$$

Table 1 Approximate and simulated critical values of ξ_2

α	$\widehat{\xi}_2$ of (12)	$\widetilde{\xi}_2$ of (13)	Simulated
0.10	0.028	0.030	0.035
0.05	0.036	0.039	0.041
0.01	0.054	0.061	0.062

and

$$\min(u, v) - uv = 2\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} \sin(j\pi u)\sin(j\pi v).$$

□

Next we propose two approximations of the critical values of ξ_2. We can show that

$$E(\xi_2) = \frac{1}{60} \quad \text{and} \quad \sigma_2^2 = Var(\xi_2) = \frac{1}{8100}.$$

Let $\xi_{2,\alpha}$ be the $(1-\alpha)^{\underline{th}}$ percentile of ξ_2. Following Lombard (1987) we only use the first term of (10) and modify it to ensure that it has the same mean as ξ_2. This gives the first approximation

$$\xi_{2,\alpha} \simeq \widehat{\xi}_{2,\alpha} = \left\{\frac{1}{6\pi^2} - \frac{1}{\pi^4}\right\}\left(\chi_{1,\alpha}^2 - 1\right) + \frac{1}{60}, \tag{12}$$

where $\chi_{1,\alpha}^2$ is the $(1-\alpha)$th percentile of the χ_1^2 distribution. Alternatively, if we ensure that the first term of (10) has the same mean and variance as ξ_2 we obtain the second approximation

$$\xi_{2,\alpha} \simeq \widetilde{\xi}_{2,\alpha} = \frac{1}{90\sqrt{2}}\left(\chi_{1,\alpha}^2 - 1\right) + \frac{1}{60}. \tag{13}$$

In Table 1 we give $\widehat{\xi}_{2,\alpha}$ of (12), $\widetilde{\xi}_{2,\alpha}$ of (13) and the simulated critical values of ξ_2. Note that the results of Table 1 are close to each other.

For $k \geq 3$, it is difficult to obtain a representation of ξ_k similar to that of (10). However, we can follow (13) to obtain the following approximate critical values of ξ_k

$$\xi_{k,\alpha} \simeq \widetilde{\xi}_{k,\alpha} = \frac{\sigma_k}{\sqrt{2}}\left(\chi_{1,\alpha}^2 - 1\right) + E(\xi_k), \tag{14}$$

where

$$E(\xi_k) = \frac{k}{(2k+1)!} \quad \text{and} \quad \sigma_k^2 = Var(\xi_k). \tag{15}$$

Note that the relations (12), (13) and (14) are true with probability close to one.

To find σ_k^2 we need to obtain $E(\xi_k^2)$. Note that

$$
\begin{aligned}
E(\xi_k^2) &= \frac{2}{((2k-1)!)^2}\int_0^1\int_0^s E\left\{B^2(t)B^2(s)\right\}dtds \\
&\quad - \frac{2}{(2k-1)!}\int_0^1\int_0^1\int_0^s Q_k(t,s)E\left\{B^2(u)B(s)B(t)\right\}dtdsdu \\
&\quad + \int_0^1\int_0^s\int_0^1\int_0^v Q_k(t,s)Q_k(u,v)E\left\{B(s)B(t)B(u)B(v)\right\}dtdsdudv \\
&= \frac{2}{((2k-1)!)^2}I_{k,1} - \frac{2}{(2k-1)!}I_{k,2} + I_{k,3}.
\end{aligned} \tag{16}
$$

We can show (see p. 43 of Rencher (1998)) that

$$
E\left\{B^2(t)B^2(s)\right\} = t\,(1-s)\left\{s-t+3t\,(1-s)\right\}, \quad 0\le t\le s\le 1, \tag{17}
$$

$$
E\left\{B^2(u)B(s)B(t)\right\} = \begin{cases} u(1-s)\left\{t+2u-3ut\right\} & \text{on } C_1=\{0<u<t<s<1\} \\ 3ut(1-u)(1-s) & \text{on } C_2=\{0<t<u<s<1\} \\ t(1-u)\left\{u+2s-3su\right\} & \text{on } C_3=\{0<t<s<u<1\} \end{cases} \tag{18}
$$

and

$$
E\left\{B(t_1)B(t_2)B(t_3)B(t_4)\right\} = t_1(1-t_4)\left\{t_3+2t_2-3t_2t_3\right\}, \quad 0<t_1<t_2<t_3<t_4<1. \tag{19}
$$

By (17)

$$
I_{k,1} = \frac{1}{40}.
$$

As to $I_{k,2}$ we have

$$
I_{k,2} = I_{k,21} + I_{k,22} + I_{k,23}, \tag{20}
$$

where

$$
I_{k,2i} = \iiint_{C_i} Q_k(t,s)E\left\{B^2(u)B(s)B(t)\right\}dtdsdu, \quad i=1,2,3. \tag{21}
$$

By (18), (20) and (21) we can compute $I_{k,2}$ for any $k\ge 3$.

Table 2 $Var(\xi_k)$

k	$Var(\xi_k)$
2	$\frac{1}{8100}$
3	$\frac{1}{9172800}$
4	$\frac{1}{34978003200}$
5	$\frac{1}{334603693670400}$

As to $I_{k,3}$ we have

$$I_{k,3} = \sum_{i=1}^{6} I_{k,3i}, \tag{22}$$

where

$$I_{k,3i} = \iiiint_{A_i} Q_k(t,s)Q_k(u,v)E\left\{B(s)B(t)B(u)B(v)\right\}dtdsdudv, \quad i=1,2,\ldots,6, \tag{23}$$

$$\begin{aligned}
A_1 &= \{0 < t < s < u < v < 1\},\\
A_2 &= \{0 < t < u < s < v < 1\},\\
A_3 &= \{0 < u < t < s < v < 1\},\\
A_4 &= \{0 < t < u < v < s < 1\},\\
A_5 &= \{0 < u < t < v < s < 1\}
\end{aligned}$$

and

$$A_6 = \{0 < u < v < t < s < 1\}.$$

By (19), (22) and (23) we can compute $I_{k,3}$ for any $k \geq 3$. In Table 2 we give the values of $\sigma_k^2 = Var(\xi_k)$ for $k = 2, 3, 4$ and 5.

We have simulated the right hand side of (6) and obtained the corresponding simulated critical values for $k = 3, \ldots, 6$. The details of this simulation are given in Sect. 5. In Table 3 we give the simulated critical values of ξ_k together with the corresponding values obtained using the proposed approximation of (14) for $\alpha = 0.01, 0.05$ and 0.10.

4 Competing tests

4.1 The rank tests of Lombard (1987)

Let $h(\cdot)$ be a real-valued and differentiable function on $(0, 1)$ and let h' be its derivative. Assume that

Table 3 Approximate and simulated critical values of ξ_k

k	α	$\widetilde{\xi}_k$	Simulated
3	0.10	9.96×10^{-4}	1.22×10^{-3}
	0.05	1.26×10^{-3}	1.42×10^{-3}
	0.10	1.91×10^{-3}	1.96×10^{-3}
4	0.10	1.75×10^{-5}	2.30×10^{-5}
	0.05	2.18×10^{-5}	2.61×10^{-5}
	0.10	3.23×10^{-5}	3.44×10^{-5}
5	0.10	1.91×10^{-7}	2.68×10^{-7}
	0.05	2.35×10^{-7}	3.00×10^{-7}
	0.10	3.43×10^{-7}	3.76×10^{-7}
6	0.10	1.43×10^{-9}	2.11×10^{-9}
	0.05	1.74×10^{-9}	2.36×10^{-9}
	0.10	2.50×10^{-9}	2.85×10^{-9}

$$\mu = \int_0^1 h(t)dt \tag{24}$$

and

$$\sigma^2 = 2\int_0^1 \int_0^y h'(x)h'(y)x(1-y)dxdy. \tag{25}$$

For $j = 1, 2, \ldots, n$, let r_j be the rank of X_j among the $X's$, $R_j = \sum_{i=1}^{j} h(\frac{r_i}{n})$ and $R_j^* = \left(R_j - j\mu\right)$. The m tests of Lombard (1987) are given by

$$m_n(k) = n^{-k-1}\sigma^{-2}\int \cdots \int_{\underline{s}} \sum_{j=1}^{k+1} \left(R^*_{[ns_j]} - R^*_{[ns_{j-1}]}\right)^2 d\underline{s} \tag{26}$$

for $k = 2, 3, \ldots$. Lombard (1987) proved that under H_0 of (1)

$$m_n(k) \xrightarrow{D} m(k) = \int \cdots \int_{\underline{s}} \sum_{j=1}^{k+1} \left(B(s_j) - B(s_{j-1})\right)^2 d\underline{s}. \tag{27}$$

By (27) and routine but tedious computations we obtain

$$m(k) = \frac{2}{(k-1)!}\int_0^1 B^2(t)dt - \int_0^1 \int_0^s Q_k^*(t,s)B(s)B(t)dtds, \tag{28}$$

Table 4 $Var(m(k))$

k	$Var(m(k))$
2	$\frac{13}{360}$
3	$\frac{1031}{226800}$
4	$\frac{131}{453600}$
5	$\frac{61307}{5448643200}$

where

$$Q_k^*(t,s) = \frac{2(1+t-s)^{k-2}}{(k-2)!} = \sum_{j=1}^{k-1} \frac{2t^{j-1}(1-s)^{k-j-1}}{(j-1)!(k-j-1)!} \quad \text{for } t < s. \tag{29}$$

Let $\gamma_{k,\alpha}$ be the $(1-\alpha)$th percentile of $m(k)$. We can follow (14) to obtain

$$\gamma_{k,\alpha} \simeq \widetilde{\gamma}_{k,\alpha} = \frac{\sigma_k^*}{\sqrt{2}}\left(\chi_{1,\alpha}^2 - 1\right) + E(m(k)), \tag{30}$$

where

$$E(m(k)) = \frac{1}{(k-1)!(k+2)} \tag{31}$$

and

$$\sigma_k^{*^2} = Var(m(k)). \tag{32}$$

The computation of $\sigma_k^{*^2}$ is parallel to the that of σ_k^2 of (15). In Table 4 we give the values of $\sigma_k^{*^2} = Var(m(k))$ for $k = 2, 3, 4$ and 5.

In Table 5 we give the simulated critical values of $m(k)$ together with the corresponding values obtained using the approximation of (30). Note that the asymptotic critical values for $m(2)$ and $m(3)$ obtained by Lombard (1987) are very close to the corresponding values of Table 5.

4.2 The rank tests of Aly and BuHamra (1996)

Following Aly and BuHamra (1996) we consider the test statistic

$$L_n(k) = n^{-k-1}\sigma^{-2}\left(\prod_{i=1}^{k+1} d_{i,n}\right)\int \cdots_{\underline{s}} \int \left\{\sum_{j=1}^{k+1} \frac{\left(R_{[ns_j]} - R_{[ns_{j-1}]}\right)^2}{d_{j,n}} - n\mu^2\right\} d\underline{s}, \tag{33}$$

Table 5 Approximate and simulated critical values of $m(k)$

k	α	$\widetilde{\gamma}_k$	Simulated
2	0.10	0.479	0.527
	0.05	0.632	0.652
	0.10	1.010	1.293
3	0.10	0.181	0.205
	0.05	0.236	0.254
	0.10	0.369	0.377
4	0.10	4.83×10^{-2}	5.64×10^{-2}
	0.05	6.19×10^{-2}	6.92×10^{-2}
	0.10	9.55×10^{-2}	9.80×10^{-2}
5	0.10	1.00×10^{-2}	1.21×10^{-2}
	0.05	1.27×10^{-2}	1.45×10^{-2}
	0.10	1.93×10^{-2}	2.04×10^{-2}
6	0.10	1.70×10^{-3}	2.15×10^{-3}
	0.05	2.14×10^{-3}	2.50×10^{-3}
	0.10	3.22×10^{-3}	3.48×10^{-3}

where μ and σ^2 are as in (24) and (25), respectively. Aly and BuHamra (1996) studied the case when $k = 2$ in (33). Let ξ_k be as in (6). It can be argued that under H_0 of (1),

$$L_n(k) \xrightarrow{D} \xi_k.$$

4.3 The cusum-type tests of Orasch (1999)

Define the test process

$$\Gamma_n(\underline{s}) = n^{-\frac{3}{2}}\delta^{-\frac{1}{2}} \left\{ \sum_{i=1}^{k} \left([ns_{i+1}] - [ns_{i-1}]\right) S_{[ns_i]} - [ns_k]S_n \right\}, \tag{34}$$

where δ is as in (5). Orasch (1999) proposed the test statistic

$$t_n(k) = \sup_{\underline{s}} \left|\Gamma_n(\underline{s})\right| \tag{35}$$

for testing H_0 of (1) against H_1 of (2). We can show that

$$\Gamma_n(\underline{s}) = n^{-\frac{3}{2}}\delta^{-\frac{1}{2}} \left\{ \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} d_{i,n}d_{j,n} \left(\overline{X}_i - \overline{X}_j\right) \right\}. \tag{36}$$

This implies that the process $\Gamma_n(\underline{s})$ of (34) and also (36) is more suitable for developing tests which are consistent against the ordered multiple k-change points alternative

$$\begin{aligned} H_{12}\ &:\ \exists 0<\lambda_1<\lambda_2<\cdots<\lambda_k<1 \quad \text{such that} \\ \mu_1 &= \cdots = \mu_{[n\lambda_1]} > \mu_{[n\lambda_1]+1} \\ &\quad = \cdots = \mu_{[n\lambda_2]} > \cdots > \mu_{[n\lambda_k]+1} = \cdots = \mu_n. \end{aligned} \tag{37}$$

In this regard we suggest the test statistics

$$t^*_{n,1}(k) = \sup_{\underline{s}} \Gamma_n(\underline{s}) \tag{38}$$

and

$$t^*_{n,2}(k) = \int \underset{\underline{s}}{\cdots} \int \Gamma_n(\underline{s}) d\underline{s} \tag{39}$$

for testing H_0 of (1) against H_{12} of (37).

Orasch (1999) argued that

$$\Gamma_n(\underline{s}) \xrightarrow{D} \Gamma(\underline{s}) = \sum_{i=1}^{k} (s_{i+1} - s_{i-1})\, W(s_i) - s_k W(1), \tag{40}$$

where $W(\cdot)$ is a Brownian motion. It is easy to show that

$$\Gamma(\underline{s}) \overset{D}{=} \Psi(\underline{s}) = \sum_{j=1}^{k} \left(s_{j+1} - s_{j-1}\right) B(s_j). \tag{41}$$

4.4 The tests of Aly et al. (2003)

For testing H_0 of (1) against H_{12} of (37), Aly et al. (2003) proposed the two tests

$$A_n(k) := \max_{\underline{s}} \sqrt{12} U_n(\underline{s}) \tag{42}$$

and

$$A^*_n(k) := \int \underset{\underline{s}}{\cdots} \int \sqrt{12} U_n(\underline{s}) d\underline{s}, \tag{43}$$

where

$$U_n(\underline{s}) = n^{-\frac{3}{2}} \sum_{i=1}^{k} \sum_{j=i}^{k} \left\{ \sum_{r=[ns_{i-1}]+1}^{[ns_i]} \sum_{l=[ns_j]+1}^{[ns_{j+1}]} I\left(X_r < X_l\right) - \frac{1}{2} d_{i,n} d_{j+1,n} \right\}.$$

Let $\Psi(\underline{s})$ be as in (41). Aly et al. (2003) proved that

$$\Gamma_n(\underline{s}) \xrightarrow{D} \Psi(\underline{s}). \tag{44}$$

By (40), (41), (44) and the results of Aly et al. (2003) we have

$$t^*_{n,2}(k) \xrightarrow{D} \tau(k) = \int_0^1 \varphi_k(t)B(t)dt$$

and

$$A^*_n(k) \xrightarrow{D} \tau(k),$$

where

$$\varphi_k(x) = \frac{1}{k!}\left\{1 - 2x^k\left(2\left[\frac{k+1}{2}\right] - k + (-1)^k\right)\right\} + \sum_{j=1}^{\left[\frac{k+1}{2}\right]+1} \frac{(-1)^{j+1}\left\{x^j + (-1)^k x^{k-j}\right\}}{j!(k-j)!}.$$

Hence, for $k = 2, 3, \ldots$

$$\tau(k) \stackrel{d}{=} N(0, \eta_k^2),$$

where

$$\eta_k^2 = 2\int_0^1 (1-y)\varphi_k(y)\int_0^y x\varphi_k(x)dxdy. \tag{45}$$

For example, $\eta_2^2 = 4.0873 \times 10^{-2}$, $\eta_3^2 = 5.9359 \times 10^{-3}$ and $\eta_4^2 = 4.2594 \times 10^{-4}$.

5 Monte Carlo studies

5.1 Asymptotic critical values

We conducted Monte Carlo studies to simulate the critical values of ξ_k and $m(k)$. We generated 2,000 realizations of the Brownian bridge $B(\cdot)$ on a grid of 2,000 points on [0,1] by generating multivariate Normal variates $\mathbf{Z}_i = (Z_{1,i}, Z_{2,i}, \cdots, Z_{2000,i})$ with covariance function, $Cov(Z_{l,i}, Z_{j,i}) = t_l(1-t_j)$, $0 < l < j \leq 2000$, where $t_l = l/(2001)$, $l = 1, \ldots, 2000$. For $i = 1, \ldots, 2000$ we computed

$$\xi_k(i) = \frac{1}{(2k-1)!} \times \frac{1}{2000}\sum_{j=1}^{2001} Z_{j,i}^2 - \frac{1}{2000^2}\sum\sum_{0<l<j\leq 2000} Q_k\left(\frac{l}{2001}, \frac{j}{2001}\right) Z_{l,i}Z_{j,i}$$

and

$$m(k,i) = \frac{2}{(k-1)!} \times \frac{1}{2000}\sum_{j=1}^{2001} Z_{j,i}^2 - \frac{1}{2000^2}\sum\sum_{0<l<j\le 2000} Q_k^*\left(\frac{l}{2001},\frac{j}{2001}\right) Z_{l,i}Z_{j,i}.$$

The simulated critical values of ξ_k of Tables 1 and 3 (resp. of $m(k)$ of Table 5) are the $(1-\alpha)$th percentiles of the $\xi_k\ (i)'\,s$ (resp. the $m(k,i)'s$).

5.2 Monte Carlo power results

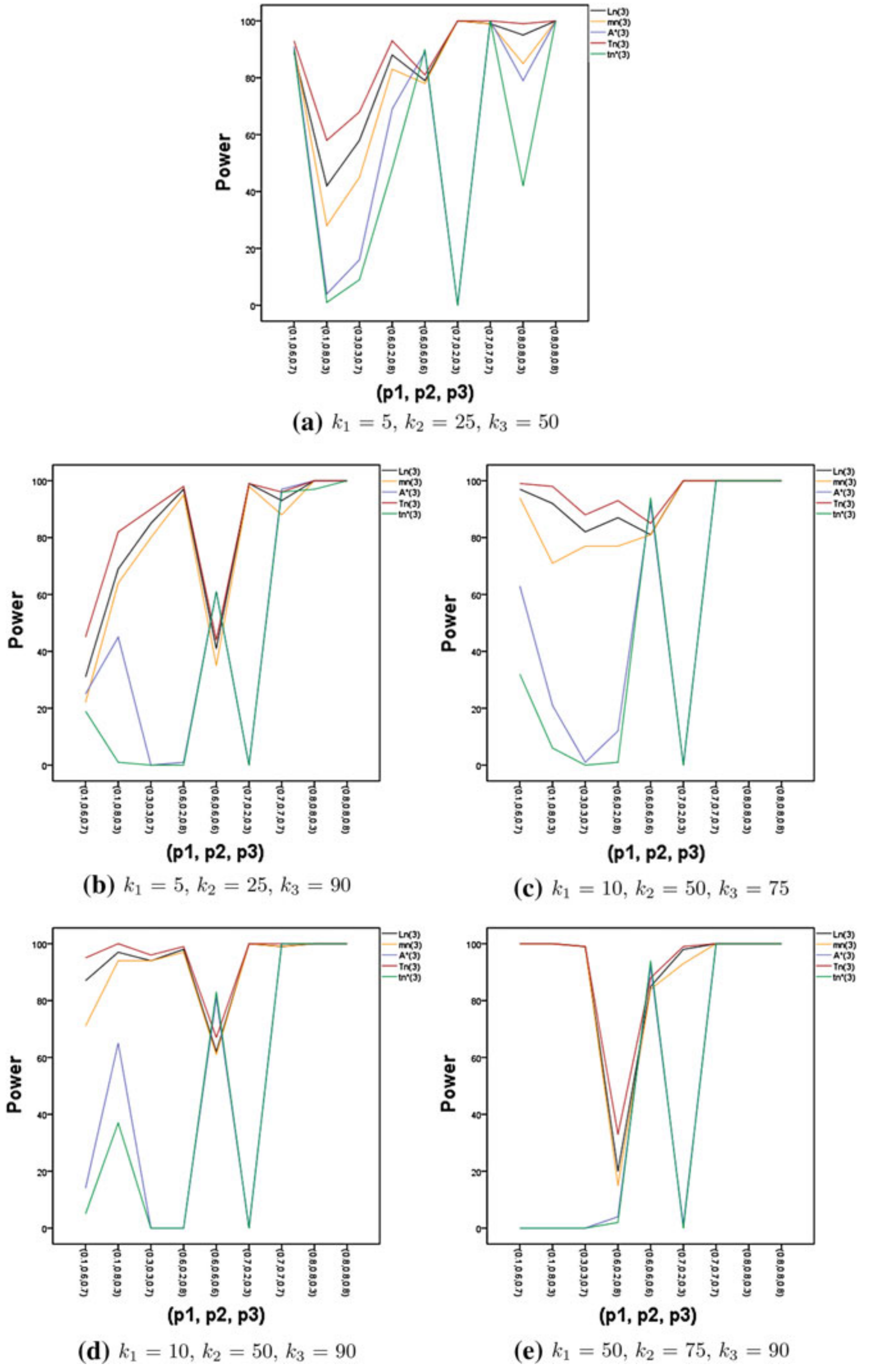
To compare the power of the proposed test of (3) with some multiple change point tests, we carried out a comprehensive simulation study. In this study we obtained Monte Carlo powers of the proposed test and the following four multiple change point tests when $k = 3$.

1. The rank test of Aly and BuHamra (1996) of (33).
2. The rank test of Lombard (1987) of (26).
3. The test of (39) of Orasch (1999).
4. The rank test $A^*(3)$ of Aly et al. (2003) of (43).

Note that the tests of Aly et al. (2003) and Orasch (1999) are consistent against H_{12} of (37).

In the Monte Carlo power study we used samples of size $n = 100$ from the Normal and Double-Exponential distributions. We employed the five change points combinations (k_1, k_2, k_3) : $(5, 25, 50), (5, 25, 90), (10, 50, 75), (10, 50, 90)$ and $(50, 75, 90)$ reflecting early, in the middle and late changes. The location parameter of X_1 is taken equal to zero and the sizes of the location shifts Δ_i at $k_i + 1,\ \ i = 1, 2, 3$, are the solutions of the equations $P(X_{k_i+1} > X_{k_i}) = p_i$, $i = 1, 2, 3$. For (p_1, p_2, p_3) we used the following combinations: (0.1, 0.6, 0.7), (0.1, 0.8, 0.3), (0.3, 0.3, 0.7), (0.6, 0.2, 0.8), (0.6, 0.6, 0.6), (0.7, 0.2, 0.3), (0.7, 0.7, 0.7), (0.8, 0.8, 0.3) and (0.8, 0.8, 0.8). Note that $p_i > 0.5$ means an upward change and $p_i < 0.5$ means a downward change.

We simulated the 0.05 critical values of the five tests and used them in the power study. We generated 2,000 samples under the alternative hypothesis and computed the fraction of times the null hypothesis was rejected for each test. As in Aly and BuHamra (1996), we also noticed that the power results of the Normal distribution are slightly lower than those of the double-exponential distribution. These results are summarized in Tables 6, 7, 8, 9 and 10 of Appendix 1 and are presented in Fig. 1 for the Normal distribution. The power results clearly suggest that, in terms of power, our proposed test performs well compared with the other tests in all considered cases. It can also be seen that when the changes are ordered, i.e., when $p_i > 0.5,\ \ i = 1, 2, 3$, the test of Aly et al. (2003) of (43) and the test of Orasch (1999) of (39) are more powerful than the other tests. Note that all the five tests have higher powers when the 3 change points are close to the middle part of the sample (see Fig. 1c).



(a) $k_1 = 5,\ k_2 = 25,\ k_3 = 50$

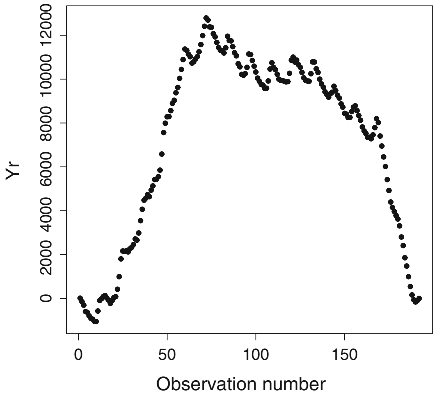
(b) $k_1 = 5,\ k_2 = 25,\ k_3 = 90$

(c) $k_1 = 10,\ k_2 = 50,\ k_3 = 75$

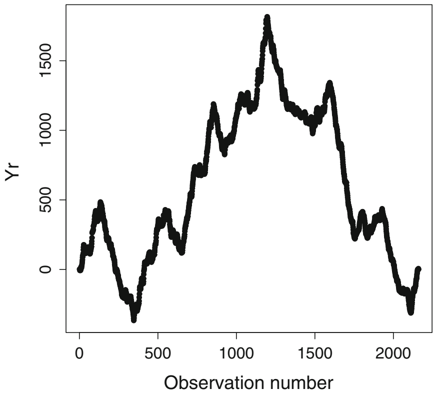
(d) $k_1 = 10,\ k_2 = 50,\ k_3 = 90$

(e) $k_1 = 50,\ k_2 = 75,\ k_3 = 90$

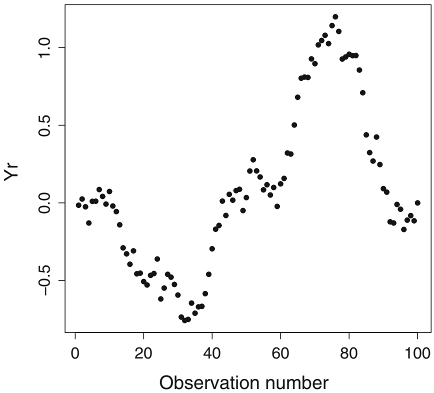
Fig. 1 Powers of test statistics $L_n(3)$, $m_n(3)$, $A^*(3)$, $T_n(3)$, $t^*_{n,2}(3)$ for $n = 100$



(a) Seat Belt data



(b) NO_2 data



(c) Lombard data

Fig. 2 Cusum plot

6 Real examples

6.1 Seat belt data

In this section we illustrate the proposed test on a road casualties data in the United Kingdom. Harvey and Durbin (1986) analyzed a data set giving the monthly totals of car drivers in UK killed or seriously injured from January, 1969 till December, 1984. Zeileis et al. (2003) analyzed the same data and concluded that the data involved two change points: the first in October, 1970 and the second in January, 1983. They also mentioned that the first change point was due to the petrol rationing and the introduction of lower speed limits during the first oil crisis. The second change point was associated with the law of compulsory wearing of seat belts which was introduced in January 31, 1983. Figure 2a displays the cumulative sum (Y_r) plot of the data. We examined the same data using our proposed test statistic. We find $T_n(2) = 0.296$, which, from Table (1), is significant at the 5 % level.

6.2 Nitrogen dioxide concentrations data

Nitrogen dioxide (NO_2) is an important traffic related air pollutant. It contributes to the formation of photochemical smog, which can have significant impacts on human health. Most of the NO_2 in cities comes from motor vehicle exhaust. Nitric oxide (NO) is emitted directly from exhausts and quickly goes on to react with ozone (O_3) to form NO_2. We consider the concentrations of NO_2 in Trafalgar Road in Greenwich (Greenwich 5). The NO_2 measurements are daily means from January first, 2000 till December 31st, 2005. The full data set is available on the London Air Quality Network (LAQN) website. Figure 2b displays the cumulative sum (Y_r) plot of the data. Carslaw and Carslaw (2007) analyzed this data and found two change points on April 10, 2001 and November 9, 2002. Also, they discussed the factors which may have contributed to the two change points. Using the proposed test we find $T_n(2) = 0.281$ which is significant at the 5 % level.

6.3 Lombard data

Lombard (1987) presented and analyzed a data set which give the radii of circular indentations cut by a milling machine. A test proposed by Lombard (1987) was implemented on the data set and concluded that the data contains two change point. Also, Fig. 2c displays the cumulative sum (Y_r) plot of the data. Using the proposed test we find $T_n(2) = 0.0393$ which is significant at the 5 % level.

7 Concluding remarks

We proposed an ANOVA-type test statistic for testing no change against a multiple change in the mean. We obtained the limiting distribution of the proposed test and proved that it is consistent against the alternative hypothesis. We obtained approximations for the limiting critical values. Monte Carlo simulation studies showed that, in terms of power, our proposed test performs well compared with a number of competing tests.

For a series of observations, when we do not know the true value of the change number k, given some upper bound k^* on k, we estimate k as the argument maximum of $T_n(k)$, $k \leq k^*$.

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Appendix 1

The following five tables summarize the Monte Carlo powers for the normal (double-exponential) distributions and $n = 100$.

Table 6 Monte Carlo powers for $k_1 = 5, k_2 = 25, k_3 = 50$

P_1, P_2, P_3	$L_n(3)$	$m_n(3)$	$A^*(3)$	$T_n(3)$	$t^*_{n,2}(3)$
0.6, 0.6, 0.6	79 (100)	78 (100)	89 (100)	81 (100)	90 (100)
0.7, 0.7, 0.7	99 (100)	99 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.8	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.3	95 (100)	85 (100)	79 (100)	99 (100)	42 (89)
0.6, 0.2, 0.8	88 (100)	83 (100)	69 (99)	93 (100)	48 (88)
0.1, 0.6, 0.7	89 (100)	91 (100)	91 (100)	93 (100)	90 (100)
0.7, 0.2, 0.3	100 (100)	100 (100)	0 (0)	100 (100)	0 (0)
0.3, 0.3, 0.7	58 (99)	45 (98)	16 (47)	68 (99)	9 (12)
0.1, 0.8, 0.3	42 (96)	28 (83)	4 (13)	58 (100)	1 (0)

Table 7 Monte Carlo powers for $k_1 = 5, k_2 = 25, k_3 = 90$

P_1, P_2, P_3	$L_n(3)$	$m_n(3)$	$A^*(3)$	$T_n(3)$	$t^*_{n,2}(3)$
0.6, 0.6, 0.6	41 (94)	35 (89)	61 (98)	44 (89)	61 (94)
0.7, 0.7, 0.7	93 (100)	88 (100)	97 (100)	96 (100)	96 (100)
0.8, 0.8, 0.8	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.3	100 (100)	100 (100)	100 (98)	100 (100)	97 (56)
0.6, 0.2, 0.8	97 (100)	95 (100)	1 (1)	98 (100)	0 (0)
0.1, 0.6, 0.7	31 (83)	22 (68)	25 (73)	45 (98)	19 (27)
0.7, 0.2, 0.3	99 (100)	98 (100)	0 (0)	99 (100)	0 (0)
0.3, 0.3, 0.7	85 (100)	80 (100)	0 (0)	90 (100)	0 (0)
0.1, 0.8, 0.3	69 (100)	64 (99)	45 (89)	82 (100)	34 (70)

Table 8 Monte Carlo powers for $k_1 = 10, k_2 = 50, k_3 = 75$

P_1, P_2, P_3	$L_n(3)$	$m_n(3)$	$A^*(3)$	$T_n(3)$	$t^*_{n,2}(3)$
0.6, 0.6, 0.6	81 (100)	81 (100)	92 (100)	85 (100)	94 (100)
0.7, 0.7, 0.7	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.8	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.3	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.6, 0.2, 0.8	87 (100)	77 (100)	12 (29)	93 (100)	1 (0)
0.1, 0.6, 0.7	97 (100)	94 (100)	63 (99)	99 (100)	32 (40)
0.7, 0.2, 0.3	100 (100)	100 (100)	0 (0)	100 (100)	0 (0)
0.3, 0.3, 0.7	82 (100)	77 (100)	1 (0)	88 (100)	0 (0)
0.1, 0.8, 0.3	92 (100)	71 (100)	21 (80)	98 (100)	6 (4)

Table 9 Monte Carlo powers for $k_1 = 10, k_2 = 50, k_3 = 90$

P_1, P_2, P_3	$L_n(3)$	$m_n(3)$	$A^*(3)$	$T_n(3)$	$t^*_{n,2}(3)$
0.6, 0.6, 0.6	62 (99)	61 (99)	81 (100)	67 (98)	83 (99)
0.7, 0.7, 0.7	99 (100)	99 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.8	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.3	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.6, 0.2, 0.8	98 (100)	97 (100)	0 (0)	99 (100)	0 (0)
0.1, 0.6, 0.7	87 (100)	71 (100)	14 (58)	95 (100)	5 (0)
0.7, 0.2, 0.3	100 (100)	100 (100)	0 (0)	100 (100)	0 (0)
0.3, 0.3, 0.7	94 (100)	94 (100)	0 (0)	96 (100)	0 (0)
0.1, 0.8, 0.3	97 (100)	94 (100)	65 (99)	100 (100)	37 (68)

Table 10 Monte Carlo powers for $k_1 = 50, k_2 = 75, k_3 = 90$

P_1, P_2, P_3	$L_n(3)$	$m_n(3)$	$A^*(3)$	$T_n(3)$	$t^*_{n,2}(3)$
0.6, 0.6, 0.6	85 (100)	84 (100)	92 (100)	88 (100)	94 (100)
0.7, 0.7, 0.7	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.8	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.8, 0.8, 0.3	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
0.6, 0.2, 0.6	20 (69)	15 (41)	4 (5)	33 (92)	2 (0)
0.1, 0.6, 0.7	100 (100)	100 (100)	0 (0)	100 (100)	0 (0)
0.7, 0.2, 0.3	98 (100)	93 (100)	1 (0)	99 (100)	0 (0)
0.3, 0.3, 0.7	99 (100)	99 (100)	0 (0)	99 (100)	0 (0)
0.1, 0.8, 0.3	100 (100)	100 (100)	0 (0)	100 (100)	0 (0)

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